

Non-Linear Electrodynamics: Zeroth and First Laws of Black Hole Mechanics

D A Rasheed

DAMTP
University of Cambridge
Silver Street
Cambridge
CB3 9EW

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Abstract

The Zeroth and First Laws of Black Hole Mechanics are derived in the context of non-linear electrodynamics coupled to gravity. The Zeroth Law is shown to hold quite generally even if the Dominant Energy Condition is violated. The derivation of the First Law is discussed in detail for general matter fields coupled to gravity. The general mass variation formula obtained includes a term previously omitted in some of the literature. This is then applied to the case of non-linear electrodynamics and the usual First Law is found to hold true. As an example, Born-Infeld theory is discussed. The results are extended to include scalar fields in a very general way, including additional terms arising from the variation of the asymptotic values of the scalars.

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1 Introduction

Theories of non-linear electrodynamics were first proposed in the 1930's as an attempt to construct a classical theory of charged particles with finite self-energy [1]. In this regard the theories were successful. However, they did not remove all of the singularities associated with a point charge and non-linear electrodynamics became less popular with the introduction of QED which provided much better agreement with experiment. Now, however, non-linear electrodynamics is making a comeback as it arises naturally in open superstring theory and the theory of D-branes.

One non-linear theory in particular, Born-Infeld theory, seems to be distinguished from all the others, occurring repeatedly in many different contexts in the study of strings and branes. For example, loop calculations in open superstring theory lead to a low energy effective action of the Born-Infeld type [2]. More recent interest has focussed on type II open superstrings with boundary states obeying Dirichlet boundary conditions. The resulting D-brane low energy effective actions, which depend on a world-volume $U(1)$ vector field, are also found to be of the Born-Infeld type [3]. One of the features of Born-Infeld theory is that it imposes a maximum allowed electric field strength in terms of a fundamental parameter of the theory. Applied to string theory, this maximum electric field strength is $\frac{1}{2\pi\alpha'}$, where α' is the inverse string tension. The existence of an upper bound on the electric field strength can be interpreted in terms of the pair production of open strings, with equal and opposite charges at their ends, by a constant electric field: In the weak field limit the pair production rate for such open strings coincides with Schwinger's classical result [4]. However, in [5] it was shown that this rate diverges as the electric field strength approaches some maximum critical value of the order of the string tension $\frac{1}{\alpha'}$.

Born-Infeld electrodynamics also has the remarkable property that, despite its highly non-linear nature, the equations of motion have an exact $SO(2)$ electric-magnetic duality invariance [6]. By adding axion and dilaton fields, this invariance may be extended to $SL(2, \mathbb{R})$ S-duality, relevant to string theory, which implies a strong-weak coupling duality of such theories. Such dualities were investigated in the context of more general non-linear electrodynamic theories [7, 8] and it was found that there are in fact an infinite number of such duality invariant theories. However, Born-Infeld, along with Maxwell theory, is the only one which is known in an exact closed form.

Recently theories such as the theory of black points, with a logarithmic electromagnetic Lagrangian coupled to gravity, have been studied in an attempt to remove some of the singularities associated with a charged black hole [9]. The divergence of the energy-momentum tensor was successfully removed, although the spacetime still exhibited a curvature singularity, albeit of a weaker variety. Born-Infeld theory has also been investigated with this goal in mind [10], the results suggesting that the usual point singularity inside the event horizon is to be replaced by a ‘ball of matter’ of finite density which somehow resists the squeeze of gravity.

The exterior of such black holes/points is, at large distances, the same as the usual black holes of Einstein-Maxwell theory. Close to the black hole, however, things may be very different. The aim of this paper is to investigate the thermodynamic properties of such black holes, focussing mainly on the First Law.

The original derivation of the First Law of Black Hole Mechanics

$$dM = \frac{\kappa}{8\pi}d\mathcal{A} + \Omega_H dJ + \Phi_H dQ \quad (1.1)$$

was done using a covariant approach [11]-[13]. It applies for perturbations from one stationary axi-symmetric solution of the Einstein-Maxwell equations to another. Since then other proofs using the Hamiltonian formalism have been given which are valid for more general perturbations [14]. Covariant techniques similar to those used in the original derivation of the First Law may also be used to prove Smarr’s formula

$$M = \frac{\kappa\mathcal{A}}{4\pi} + 2\Omega_H J + \Phi_H Q. \quad (1.2)$$

However, it was later realized [15] that, in the case of Einstein-Maxwell theory, these two formulae are very closely related and that it is possible to deduce one directly from the other using the homogeneity of the mass M as a function of $\sqrt{\mathcal{A}}$, \sqrt{J} and Q . Thus it is only necessary to derive *one* of the two formulae – preferably the Smarr formula, since its derivation is much simpler.

In the case of non-linear electrodynamics, however, one no longer has homogeneity of the mass function and so, not only is it no longer possible to pass easily between these two formulae, but also one can no longer expect both to hold (indeed, a priori one has no reason to expect that either of them

holds!). The main conclusion of this paper is that the First Law does indeed continue to hold for *all* theories of non-linear electrodynamics (and hence the Smarr formula does not).

Section 2 contains a brief introduction and description of what is meant by a non-linear theory of electrodynamics. The conventions being used will be established and some useful electromagnetic quantities defined for the case of stationary axi-symmetric black holes. As a precursor to the main discussion of the First Law, in section 3 some brief comments will be made concerning the Zeroth Law. The main point is that, for non-linear electrodynamics, the requirement that the Dominant Energy Condition be satisfied is no longer necessary to prove the Zeroth Law. In section 4 the covariant approach will be used to establish the First Law for a fully general energy-momentum tensor, describing *any* form of matter fields coupled to gravity. The mass variation formula given will include a term, relevant to rotating black holes, previously omitted in [12]. In section 5 this result will then be applied to non-linear electrodynamics coupled to gravity. As an example of such a theory, the static black holes of Born-Infeld theory will be discussed in section 6. Finally, in section 7, the inclusion of scalar fields is discussed.

Throughout this paper, geometrical units will be used: $G = c = 4\pi\epsilon_0 = \frac{\mu_0}{4\pi} = 1$.

2 Non-Linear Electrodynamics

The usual (linear) theory of electrodynamics coupled to gravity (Einstein-Maxwell theory) may be described using the Lagrangian formulation. The action is

$$S = \int d^4x \sqrt{g} (R - F_{\mu\nu} F^{\mu\nu}) \quad (2.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Varying with respect to A_μ gives the electromagnetic equations of motion

$$\nabla_\mu F^{\mu\nu} = 0 \quad \text{or} \quad d \star F = 0 \quad (2.2)$$

which are linear in $F_{\mu\nu}$. This is what is meant *linear* electrodynamics. Note that the Einstein field equations obtained by varying with respect to the metric $g_{\mu\nu}$ are still of course non-linear (in both $g_{\mu\nu}$ and $F_{\mu\nu}$):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2 \left(F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F^2 \right). \quad (2.3)$$

In non-linear electrodynamics one replaces the usual electromagnetic Lagrangian $L_F = -F^2$ by a more general function of the field strength $F_{\mu\nu}$. For example, in the theory of black points [9] the Lagrangian $-\log(1 + F^2)$ is used. Since the Lagrangian is no longer quadratic in $F_{\mu\nu}$, the equations of motion are both non-linear in $F_{\mu\nu}$.

Consider a general theory of non-linear electrodynamics coupled to gravity, described by the action

$$S = \int d^4x \sqrt{g} (R + L_F) \quad (2.4)$$

where $L_F(F_{\mu\nu})$ is assumed to be function of the field strength tensor $F_{\mu\nu}$ only, not containing any higher derivative terms in $F_{\mu\nu}$. If the theory is to agree with the usual Einstein-Maxwell theory for weak fields (as is the case for the logarithmic Lagrangian above and for Born-Infeld theory which will be discussed in section 6), then L_F must be of the form

$$L_F = -F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(F^4). \quad (2.5)$$

It is useful to define the second rank tensor $G^{\mu\nu}$ by ¹

$$G^{\mu\nu} = -\frac{1}{2} \frac{\partial L_F}{\partial F_{\mu\nu}}. \quad (2.6)$$

So for Lagrangians of the form (2.5) $G^{\mu\nu}$ takes the form $G^{\mu\nu} = F^{\mu\nu} + \mathcal{O}(F^3)$. Varying the action with respect to A_μ gives the electromagnetic field equations

$$\nabla_\mu G^{\mu\nu} = 0 \quad \text{or} \quad d \star G = 0. \quad (2.7)$$

Since $G_{\mu\nu}$ is in general a highly non-linear function of $F_{\mu\nu}$, these equations are non-linear. The field strength $F_{\mu\nu}$ also satisfies the Bianchi identities

$$\nabla_{[\mu} F_{\nu\lambda]} = 0 \quad \text{or} \quad dF = 0. \quad (2.8)$$

Note that there is a clear similarity between these last two equations. If $F_{\mu\nu}$ and $G_{\mu\nu}$ were independent variables then the theory would have an $SO(2)$ symmetry corresponding to rotating F into $\star G$ (electric-magnetic duality).

¹There is some ambiguity in the definition of this partial derivative depending on whether or not one takes into account the antisymmetry of $F_{\mu\nu}$. Here $F_{\mu\nu}$ and $F_{\nu\mu}$ are treated as independent variables in the partial derivative

However, since $F_{\mu\nu}$ and $G_{\mu\nu}$ are not independent, this duality mixing the Bianchi identities with the equations of motion is in general inconsistent. In the special case of Born-Infeld theory this duality is an exact symmetry of the theory (see [7] for a more complete discussion of electric-magnetic duality in non-linear electrodynamics). Varying the action with respect to the metric $g_{\mu\nu}$ gives the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu} \quad (2.9)$$

where the energy-momentum tensor $T_{\mu\nu}$ for non-linear electrodynamics is found to be

$$T_{\mu\nu} = G_{\mu}{}^{\lambda}F_{\nu\lambda} + \frac{1}{4}g_{\mu\nu}L_F. \quad (2.10)$$

Note that L_F can be a function only of the two scalar invariants $F^2 = F_{\mu\nu}F^{\mu\nu}$ and $F \star F = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ since (in four dimensions) these are the only two scalar invariants which can be formed from $F_{\mu\nu}$. This means that $G_{\mu\nu}$ must be of the form

$$G_{\mu\nu} = aF_{\mu\nu} + b \star F_{\mu\nu} \quad (2.11)$$

for some scalar functions a and b . Here \star is the Hodge star operator given by $\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ where $\epsilon_{\mu\nu\rho\sigma}$ are the components of the covariantly constant volume 4-form. F and $\star F$ satisfy the useful identities $F_{\mu}{}^{\lambda} \star F_{\nu\lambda} = \frac{1}{4}F \star F g_{\mu\nu}$ and $\star F_{\mu}{}^{\lambda} \star F_{\nu\lambda} = -F_{\mu}{}^{\lambda}F_{\nu\lambda}$. This ensures that $T_{\mu\nu}$ as defined above is symmetric.

Consider now stationary axi-symmetric black hole solutions. Let k be the (unique) time-translation Killing vector, normalized so that $k^2 = -1$ at infinity and let m denote the rotational Killing vector whose orbits are closed curves with parameter length 2π . Thus the metric $g_{\mu\nu}$, the field strength tensor $F_{\mu\nu}$ and also $G_{\mu\nu}$ are invariant under the action of the time-translational and rotational symmetries generated by k and m :

$$\begin{aligned} \mathcal{L}_k g_{\mu\nu} &= \mathcal{L}_m g_{\mu\nu} = 0 \\ \mathcal{L}_k F_{\mu\nu} &= \mathcal{L}_m F_{\mu\nu} = 0 \\ \mathcal{L}_k G_{\mu\nu} &= \mathcal{L}_m G_{\mu\nu} = 0 \end{aligned} \quad (2.12)$$

where \mathcal{L} denotes the Lie derivative. In the usual coordinates (t, r, θ, ϕ) these Killing vectors are

$$k = \frac{\partial}{\partial t} \quad \text{and} \quad m = \frac{\partial}{\partial \phi}. \quad (2.13)$$

One may then form a new Killing vector ξ as a linear combination of k and m which is tangent to the null generators of the black hole event horizon \mathcal{H}

$$\xi = k + \Omega_H m. \quad (2.14)$$

The requirement that ξ be null on the horizon fixes the constant Ω_H which may be interpreted as the angular velocity of the event horizon.

Electric and magnetic field vectors may now be defined as follows

$$E_\mu = F_{\mu\nu} \xi^\nu \quad \text{and} \quad H_\mu = -\star G_{\mu\nu} \xi^\nu. \quad (2.15)$$

The signs are chosen so that in flat space their spatial parts agree with the usual definitions $E_i = F_{i0}$ and $H_i = \frac{1}{2} \epsilon_{ijk} G_{jk}$. Using the Bianchi identities (2.8) one has

$$\nabla_{[\mu} E_{\nu]} = -\frac{1}{2} \mathcal{L}_\xi F_{\mu\nu} = 0. \quad (2.16)$$

So E_μ may be written in terms of a (co-rotating) electric potential Φ via

$$E_\mu = \partial_\mu \Phi \quad (2.17)$$

and choosing $\Phi \rightarrow 0$ as $r \rightarrow \infty$ determines it uniquely.

Note that due to the gauge freedom the vector potential A_μ is not necessarily invariant under the stationary and axi-symmetry group actions (for example, the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu f(t)$ destroys time-translation invariance). So in general one has

$$\mathcal{L}_\xi A_\mu = \Lambda_\mu \neq 0 \quad (2.18)$$

where Λ_μ is obviously a gauge dependent quantity. Taking the exterior derivative of the above equation implies that $\nabla_{[\mu} \Lambda_{\nu]} = 0$ and hence

$$\Lambda_\mu = \partial_\mu \Lambda. \quad (2.19)$$

Then it can be shown that

$$E_\mu = \partial_\mu (A_\nu \xi^\nu - \Lambda) \quad (2.20)$$

so comparison with (2.17) gives

$$\Phi = A_\nu \xi^\nu - \Lambda + \text{const.} \quad (2.21)$$

Typically, for an electrically charged black hole, one would choose a gauge in which $\Lambda = 0$ and $A_\mu \rightarrow 0$ as $r \rightarrow \infty$. Then *in this gauge*

$$\Phi = A_\nu \xi^\nu \quad (2.22)$$

which is commonly taken as the definition of the co-rotating electric potential. However, this is not very satisfactory since it is not a gauge invariant statement. The definition (2.17) is preferable since it guarantees that Φ is gauge invariant (at least up to a constant) and also that it is preserved by the stationary and axi-symmetry group actions.

Considering magnetically charged black holes it is also possible to define a magnetic potential Ψ as follows: Using the electromagnetic field equations (2.7) one has

$$\nabla_{[\mu} H_{\nu]} = \frac{1}{2} \mathcal{L}_\xi \star G_{\mu\nu} = 0 \quad (2.23)$$

and so H_μ may be written as

$$H_\mu = \partial_\mu \Psi \quad (2.24)$$

and choosing $\Psi \rightarrow 0$ as $r \rightarrow \infty$ fixes it uniquely.

Due to the antisymmetry of $F_{\mu\nu}$ and $\star G_{\mu\nu}$, E_μ and H_μ satisfy $\xi.E = \xi.H = 0$ and hence they are tangent to the horizon \mathcal{H} . Using Raychaudhuri's equation $R_{\mu\nu} \xi^\mu \xi^\nu = 0$ on \mathcal{H} and the field equations, one has $T_{\mu\nu} \xi^\mu \xi^\nu = 0$ on \mathcal{H} . Substituting for $T_{\mu\nu}$ from (2.10) and using (2.11) this implies that

$$F_\mu{}^\lambda F_{\nu\lambda} \xi^\mu \xi^\nu = E_\mu E^\mu = 0 \quad \text{on } \mathcal{H}. \quad (2.25)$$

Similarly it is possible to deduce that H_μ is also null on the horizon:

$$H_\mu H^\mu = 0 \quad \text{on } \mathcal{H}, \quad (2.26)$$

So E_μ and H_μ must be proportional to ξ_μ on \mathcal{H} and hence the potentials Φ and Ψ take *constant* values Φ_H and Ψ_H respectively on the horizon.

3 Zeroth Law

The Zeroth Law of Black Hole Mechanics states that the surface gravity κ of a stationary black hole is constant over the event horizon. The original proof [11, 13], applicable to any Killing horizon, relies on the energy-momentum tensor satisfying the Dominant Energy Condition. More recently [16] the Zeroth Law has been proved without resort to the Dominant Energy Condition, using only the assumption that the event horizon be a bifurcate Killing horizon. In this section it will be shown that, in the case of non-linear electrodynamics, neither of these assumptions is required.

The standard proof of the Zeroth Law proceeds as follows: Define the vector J by

$$J_\mu = -T_{\mu\nu}\xi^\nu. \quad (3.1)$$

Then by Raychaudhuri's equation $\xi.J = 0$ on \mathcal{H} , so J is tangent to the horizon and hence it is spacelike or null there. At this point one appeals to the Dominant Energy Condition which implies that J must be timelike or null. Therefore J is null on the horizon and hence proportional to ξ . Now one uses the identity $\xi_{[\mu}\partial_{\nu]}\kappa = -\xi_{[\mu}R_{\nu]\lambda}\xi^\lambda$ on \mathcal{H} , which follows from the definition $\xi^\mu\nabla_\mu\xi^\nu = \kappa\xi^\nu$ on \mathcal{H} and the Frobenius condition $\xi_{[\mu}\nabla_\nu\xi_{\lambda]} = 0$ on \mathcal{H} . This identity together with the Einstein field equations gives

$$\xi_{[\mu}\partial_{\nu]}\kappa = 2\xi_{[\mu}J_{\nu]} \quad \text{on } \mathcal{H}. \quad (3.2)$$

Now the right hand side vanishes since J is proportional to ξ on \mathcal{H} . Hence $\partial_\mu\kappa$ is proportional to ξ which is normal to the horizon and so κ is constant on \mathcal{H} .

In non-linear electrodynamics, however, the use made of the Dominant Energy Condition above is unnecessary. Using (2.10) and (2.11) one may write the energy-momentum tensor in the form

$$T_{\mu\nu} = aF_\mu{}^\lambda F_{\nu\lambda} + \frac{1}{4}(L_F + bF \star F)g_{\mu\nu}. \quad (3.3)$$

Now, as was proved at the end of the last section, E_μ is proportional to ξ_μ on \mathcal{H} . Therefore J_μ , as defined above, must also be proportional to ξ_μ on \mathcal{H} , as a direct consequence of the special form of the energy-momentum tensor for non-linear electrodynamics. The Zeroth Law then follows as above. Note that this argument remains true even if the Dominant Energy Condition is violated and is valid for any type of Killing horizon.

4 First Law

In this section the mass variation formula for stationary black holes in General Relativity coupled to general matter fields will be derived in some detail, using the traditional covariant approach. The final formula obtained will correct a minor error in equation (92) of [12] which omits a term proportional to $\delta\Omega_H$.

First some conventions must be established. The Komar integral for the total mass of a stationary axi-symmetric asymptotically flat spacetime is given by

$$M = -\frac{1}{8\pi} \oint_{\infty} dS_{\mu\nu} \nabla^{\mu} k^{\nu} \quad (4.1)$$

where the integral is over a closed surface, topologically a 2-sphere, at spatial infinity. Similarly the total angular momentum may be expressed as a Komar integral:

$$J = \frac{1}{16\pi} \oint_{\infty} dS_{\mu\nu} \nabla^{\mu} m^{\nu}. \quad (4.2)$$

It is convenient to define the corresponding integral over the event horizon which may be interpreted as the horizon angular momentum:

$$J_H = \frac{1}{16\pi} \oint_{\mathcal{H}} dS_{\mu\nu} \nabla^{\mu} m^{\nu}. \quad (4.3)$$

The surface element on the event horizon is given by $dS_{\mu\nu} = 2n_{[\mu}\xi_{\nu]}d\mathcal{A}$ where n is another independent normal to the horizon satisfying $n.\xi = 1$ on \mathcal{H} . Let Σ be a spacelike hypersurface extending from \mathcal{H} out to spatial infinity. Then, with these conventions, if $X^{\mu\nu}$ is any antisymmetric tensor, Stoke's formula takes the form

$$2 \int_{\Sigma} dS_{\mu} \nabla_{\nu} X^{\mu\nu} = \oint_{\infty} dS_{\mu\nu} X^{\mu\nu} - \oint_{\mathcal{H}} dS_{\mu\nu} X^{\mu\nu}. \quad (4.4)$$

Applying this to the expressions for J and J_H and making use of the identity $\nabla_{\mu} \nabla_{\nu} m_{\rho} = R_{\rho\nu\mu\sigma} m^{\sigma}$ satisfied by any Killing vector gives

$$J - J_H = \frac{1}{8\pi} \int_{\Sigma} dS_{\mu} R^{\mu}{}_{\nu} m^{\nu}. \quad (4.5)$$

Thus for vacuum solutions $J_H = J$, the total angular momentum of the black hole. In general they differ and the difference $J - J_H$ may be interpreted as

the angular momentum of the matter fields present. Using these formulae it is straightforward to derive Smarr-type formulae for the mass:

$$M = \frac{\kappa \mathcal{A}}{4\pi} + 2\Omega_H J - \frac{1}{4\pi} \int_{\Sigma} dS_{\mu} R^{\mu}{}_{\nu} \xi^{\nu} \quad (4.6)$$

or alternatively:

$$M = \frac{\kappa \mathcal{A}}{4\pi} + 2\Omega_H J_H - \frac{1}{4\pi} \int_{\Sigma} dS_{\mu} R^{\mu}{}_{\nu} k^{\nu} \quad (4.7)$$

In Einstein-Maxwell theory the integral over Σ in (4.6) is easily evaluated giving $\Phi_H Q$ ($+\Psi_H P$ if the black hole has magnetic charge as well). However, in non-linear electrodynamics there is no easy way to evaluate the integral and a simple check for known solutions, such as the static Born-Infeld black holes discussed in section 6, shows that the simple Smarr formula from Einstein-Maxwell theory no longer holds.

Now consider a small variation of this solution to another stationary axisymmetric solution. Denote the variation of the metric $\delta g_{\mu\nu}$ by $h_{\mu\nu}$ where the indices on h may be freely raised and lowered using the metric $g_{\mu\nu}$. Then the variation of the inverse metric is given by $\delta g^{\mu\nu} = -h^{\mu\nu}$. Now there is a certain gauge freedom when comparing two different solutions which may be used to ensure that the stationary and rotational Killing vectors do not change under the variation:

$$\delta k^{\mu} = 0 \quad \text{and} \quad \delta m^{\mu} = 0. \quad (4.8)$$

The gauge freedom may also be used to ensure that the horizon \mathcal{H} is in the same position in both solutions which is consistent with (4.8) since the horizon is a fixed point set of k and m . Then the variation of ξ^{μ} is given by

$$\delta \xi^{\mu} = \delta \Omega_H m^{\mu}. \quad (4.9)$$

Since the horizon is in the same position in both solutions the vectors normal to it in each solution must be parallel. So $\delta \xi_{\mu} = f \xi_{\mu}$ and $\delta n^{\mu} = g n^{\mu}$ on \mathcal{H} for some functions f and g . Preservation of the normalization condition $n \cdot \xi = 1$ implies that $f + g = 0$. This then gives the useful relation

$$n^{\mu} \delta \xi_{\nu} + \delta n^{\mu} \xi_{\nu} = 0 \quad \text{on } \mathcal{H}. \quad (4.10)$$

Another useful relation comes from the fact that the variation preserves time-translational and rotational invariance so that

$$\mathcal{L}_\xi \delta \xi_\mu = \xi^\nu \nabla_\nu \delta \xi_\mu + \delta \xi_\nu \nabla_\mu \xi^\nu = 0. \quad (4.11)$$

The next step is to calculate the variation of the surface gravity κ . The most convenient definition of κ to use for this purpose is

$$\kappa = -\frac{1}{2} n^\mu \nabla_\mu (\xi^\nu \xi_\nu) \quad \text{on } \mathcal{H}. \quad (4.12)$$

This then gives

$$\delta \kappa = -\frac{1}{2} (n^\mu \xi^\nu + n^\nu \xi^\mu) \nabla_\mu \delta \xi_\nu - \delta \Omega_H n_\mu \xi_\nu \nabla^\mu m^\nu \quad \text{on } \mathcal{H} \quad (4.13)$$

where use has been made of (4.9), (4.10), (4.11) and the fact that the time-translational and rotational group actions commute $[k, m] = 0$. Now using the fact that $\delta \xi_\mu$ is parallel to ξ_μ , the first term on the right hand side may be simplified to give $-\frac{1}{2} \nabla^\mu \delta \xi_\mu$. But $\delta \xi_\mu = h_{\mu\nu} \xi^\nu + \delta \Omega_H m_\mu$ and so the variation of the surface gravity simplifies giving

$$\delta \kappa = -\frac{1}{2} \xi^\mu \nabla^\nu h_{\mu\nu} - \delta \Omega_H n_\mu \xi_\nu \nabla^\mu m^\nu \quad \text{on } \mathcal{H}. \quad (4.14)$$

It is now necessary to integrate this over the horizon \mathcal{H} in order to derive the First Law. With some foresight, the first term on the right hand side of the above expression can be put into the more convenient form:

$$-\frac{1}{2} \xi^\mu \nabla^\nu h_{\mu\nu} = -(n_\mu \xi_\nu - n_\nu \xi_\mu) \xi^\mu \nabla^{[\lambda} h^{\nu]}_\lambda \quad \text{on } \mathcal{H} \quad (4.15)$$

where use has been made of the fact that the variation is to another stationary axi-symmetric solution and hence $\mathcal{L}_\xi h = 0$. Thus

$$\delta \kappa d\mathcal{A} = -\frac{1}{2} dS_{\mu\nu} \left(2\xi^\mu \nabla^{[\lambda} h^{\nu]}_\lambda + \delta \Omega_H \nabla^\mu m^\nu \right) \quad \text{on } \mathcal{H} \quad (4.16)$$

and integrating over \mathcal{H} gives

$$\mathcal{A} \delta \kappa = -8\pi \delta \Omega_H J_H - \oint_{\mathcal{H}} dS_{\mu\nu} \xi^\mu \nabla^{[\lambda} h^{\nu]}_\lambda \quad (4.17)$$

since κ is constant over \mathcal{H} by the Zeroth Law. Using Stoke's formula this last integral can be converted to an integral at spatial infinity and an integral over

Σ . The integral over spatial infinity may be evaluated using the asymptotic form of the metric:

$$\oint_{\infty} dS_{\mu\nu} \xi^{\mu} \nabla^{[\lambda} h^{\nu]}_{\lambda} = 4\pi\delta M. \quad (4.18)$$

So equation (4.17) becomes

$$4\pi\delta M + \mathcal{A}\delta\kappa + 8\pi\delta\Omega_H J_H = \int_{\Sigma} dS_{\mu} \xi^{\mu} \nabla_{\nu} \nabla^{[\lambda} h^{\nu]}_{\lambda} \quad (4.19)$$

where again use has been made of the fact that $\mathcal{L}_{\xi} h_{\mu\nu} = 0$ in rearranging the final integral. Now the integrand in this expression is related to the variation of the Ricci scalar R under metric perturbations:

$$\delta R = 2\nabla_{\nu} \nabla^{[\lambda} h^{\nu]}_{\lambda} - h^{\mu\nu} R_{\mu\nu}. \quad (4.20)$$

So using this and also making use of (4.5) and (4.9) to eliminate J_H in favour of J in (4.19) gives

$$4\pi\delta M + \mathcal{A}\delta\kappa + 8\pi\delta\Omega_H J = \int_{\Sigma} dS_{\mu} \left\{ \frac{1}{2} \xi^{\mu} (\delta R + h^{\rho\sigma} R_{\rho\sigma}) + R^{\mu}_{\nu} \delta \xi^{\nu} \right\}. \quad (4.21)$$

Using the Einstein equations and the fact that $\delta dS_{\mu} = \frac{1}{2} h dS_{\mu}$ where $h = h_{\mu\nu} g^{\mu\nu}$ equation (4.21) may be rearranged to give

$$4\pi\delta M + \mathcal{A}\delta\kappa + 8\pi\delta\Omega_H J = \int_{\Sigma} dS_{\mu} (\xi^{\mu} T_{\rho\sigma} h^{\rho\sigma} + 2T^{\mu}_{\nu} \delta \xi^{\nu}) + \frac{1}{2} \delta \int_{\Sigma} dS_{\mu} \xi^{\mu} R. \quad (4.22)$$

Finally, combining this with the variation of the Smarr formula (4.6) and using the Einstein equations again gives the First Law of Black Hole Mechanics for stationary axi-symmetric black holes in General Relativity coupled to arbitrary matter fields with energy-momentum tensor $T_{\mu\nu}$:

$$\begin{aligned} \delta M = & \frac{\kappa}{8\pi} \delta \mathcal{A} + \Omega_H \delta J + \frac{1}{8\pi} \int_{\Sigma} dS_{\mu} \xi^{\mu} T_{\rho\sigma} h^{\rho\sigma} - \frac{1}{4\pi} \delta \int_{\Sigma} dS_{\mu} T^{\mu}_{\nu} \xi^{\nu} \\ & + \frac{1}{4\pi} \delta \Omega_H \int_{\Sigma} dS_{\mu} T^{\mu}_{\nu} m^{\nu}. \end{aligned} \quad (4.23)$$

In the case where the matter fields may be described by a Lagrangian L_F , the energy-momentum tensor $T_{\mu\nu}$ is related to the variation of the Lagrangian with respect to the metric:

$$\delta_g (\sqrt{g} L_F) = 2\sqrt{g} T_{\mu\nu} h^{\mu\nu}. \quad (4.24)$$

Here δ_g denotes variation with respect to $g_{\mu\nu}$ and δ_A will denote variation with respect to the matter fields and so $\delta = \delta_g + \delta_A$. Using these relations the First Law may be rewritten in the form

$$\begin{aligned} \delta M = & \frac{\kappa}{8\pi} \delta \mathcal{A} + \Omega_H \delta J - \frac{1}{4\pi} \delta \int_{\Sigma} dS_{\mu} \left(T^{\mu}{}_{\nu} \xi^{\nu} - \frac{1}{4} L_F \xi^{\mu} \right) - \frac{1}{16\pi} \delta_A \int_{\Sigma} dS_{\mu} \xi^{\mu} L_F \\ & + \frac{1}{4\pi} \delta \Omega_H \int_{\Sigma} dS_{\mu} \left(T^{\mu}{}_{\nu} m^{\nu} - \frac{1}{4} L_F m^{\mu} \right). \end{aligned} \quad (4.25)$$

Note that by using equation (4.5), this statement of the First Law may be written in terms of δJ_H rather than δJ . In this case ξ is replaced by k in the first two integrals of (4.25) and the final integral is absent. This then agrees with equation (90) of [12]. However, in going from equation (90) to (92) in [12], the variation of Ω_H was omitted. Thus the final term in (4.25) corrects equation (92) of [12], although their derivation up to that point remains correct. For the purposes of this paper, the form of the First Law (4.25), in terms of ξ and δJ , is more convenient because the electric and magnetic potentials Φ and Ψ must be defined in terms of ξ (not k) so that they are constant on the horizon.

The last term in equation (4.25) has an interesting interpretation. It may be regarded as the contribution to the mass variation due to the angular momentum of the matter fields outside the event horizon. For example, if the surface Σ is chosen so that it is invariant under the rotational symmetry, i.e. $dS_{\mu} m^{\mu} = 0$, then this last term evaluates to $(J - J_H) \delta \Omega_H$. Note that in all the calculations above, no specific assumptions have been made about the surface Σ other than the fact that it has a boundary on \mathcal{H} and extends out to spatial infinity. Since, in general, the integrands in the above formula are not total derivatives, the integrals will depend on the choice of Σ .

In the next section the case of non-linear electrodynamics will be discussed. In this case the integrand of the final integral in (4.25) is a total derivative and so the integral is independent of the particular choice of Σ . So Σ may indeed be chosen to be invariant under the rotational symmetry, to simplify the evaluation of the integral. Thus, in this case, the final integral gives the angular momentum of the electromagnetic field, $J_F = J - J_H$ and moreover this result is independent of the choice of Σ .

5 Application to Non-Linear Electrodynamics

Using the special form of the energy-momentum tensor for non-linear electrodynamics (2.10) the First Law may be written as

$$\delta M = \frac{\kappa}{8\pi} \delta \mathcal{A} + \Omega_H \delta J - \delta I_1 - \delta_A I_2 + \delta \Omega_H I_3 \quad (5.1)$$

where

$$\begin{aligned} I_1 &= \frac{1}{4\pi} \int_{\Sigma} dS_{\mu} G^{\mu\lambda} F_{\nu\lambda} \xi^{\nu}, \\ I_2 &= \frac{1}{16\pi} \int_{\Sigma} dS_{\mu} \xi^{\mu} L_F, \\ I_3 &= \frac{1}{4\pi} \int_{\Sigma} dS_{\mu} G^{\mu\lambda} F_{\nu\lambda} m^{\nu}. \end{aligned} \quad (5.2)$$

Using (2.15) and (2.17) together with the electromagnetic equations of motion (2.7) enables the integrand in I_1 to be written as a total derivative:

$$I_1 = -\frac{1}{4\pi} \int_{\Sigma} dS_{\mu} \nabla_{\nu} (\Phi G^{\mu\nu}). \quad (5.3)$$

Using Stoke's formula and the asymptotic condition $\Phi \rightarrow 0$ as $r \rightarrow \infty$, I_1 may be expressed as an integral over the horizon. Now the electric charge of the black hole is given by

$$Q = -\frac{1}{8\pi} \oint dS_{\mu\nu} G^{\mu\nu} \quad (5.4)$$

where the integral may be taken over any closed 2-surface enclosing the charge and is independent of the particular surface chosen by virtue of the equations of motion (2.7). Therefore, using the fact that Φ is constant on the horizon, one finds that

$$I_1 = -\Phi_H Q. \quad (5.5)$$

Evaluation of the second integral I_2 is not possible, in general, since the precise form of L_F is not known. However, in the calculation of the First Law it is not necessary to evaluate I_2 , only its variation with respect to A_{μ} .

This is done with the help of the definition of $G^{\mu\nu}$ in terms of the variation of L_F with respect to $F_{\mu\nu}$ which gives

$$\delta_A L_F = \frac{\partial L_F}{\partial F_{\mu\nu}} \delta_A F_{\mu\nu} = -2G^{\mu\nu} \delta_A F_{\mu\nu} \quad (5.6)$$

and so

$$\delta_A I_2 = -\frac{1}{8\pi} \int_{\Sigma} dS_{\mu} \xi^{\mu} G^{\rho\sigma} \delta_A F_{\rho\sigma}. \quad (5.7)$$

In order to express the integrand in (5.7) in a more convenient form, consider the vector

$$Y^{\mu} = \epsilon^{\mu\nu\rho\sigma} (\delta_A E_{\nu} \star G_{\rho\sigma} - H_{\nu} \delta_A F_{\rho\sigma}). \quad (5.8)$$

By using the definitions of E_{μ} and H_{μ} (2.15) and expanding out products of ϵ tensors, Y^{μ} becomes

$$Y^{\mu} = \xi^{\mu} G^{\rho\sigma} \delta_A F_{\rho\sigma} \quad (5.9)$$

which is precisely the integrand in (5.7). Alternatively, using $\star\star = -1$, Y^{μ} may be written as

$$Y^{\mu} = -2 (\delta_A E_{\nu} G^{\mu\nu} + H_{\nu} \delta_A \star F^{\mu\nu}). \quad (5.10)$$

Now from equations (2.15) and (4.9) the variation of E_{ν} is

$$\delta_A E_{\nu} = (\delta - \delta_g) E_{\nu} = \delta E_{\nu} - \delta \Omega_H F_{\nu\lambda} m^{\lambda} \quad (5.11)$$

and so

$$\begin{aligned} Y^{\mu} &= -2 (\delta E_{\nu} G^{\mu\nu} + H_{\nu} \delta_A \star F^{\mu\nu}) + 2\delta \Omega_H G^{\mu\nu} F_{\nu\lambda} m^{\lambda} \\ &= -2\nabla_{\nu} (\delta \Phi G^{\mu\nu} + \Psi \delta_A \star F^{\mu\nu}) + 2\delta \Omega_H G^{\mu\nu} F_{\nu\lambda} m^{\lambda} \end{aligned} \quad (5.12)$$

using (2.17), (2.24), the Bianchi identities (2.8) and the equations of motion (2.7) to obtain the last line. Thus using Stoke's formula and the asymptotic conditions $\Phi, \Psi \rightarrow 0$ as $r \rightarrow \infty$ the variation of I_2 may be written as an integral over the horizon plus an integral over Σ which can be seen to be proportional to the third integral I_3 :

$$\delta_A I_2 = -\frac{1}{8\pi} \oint_{\mathcal{H}} dS_{\mu\nu} (\delta \Phi G^{\mu\nu} + \Psi \delta_A \star F^{\mu\nu}) + \delta \Omega_H I_3. \quad (5.13)$$

Now defining the magnetic charge P as

$$P = \frac{1}{8\pi} \oint dS_{\mu\nu} \star F^{\mu\nu} \quad (5.14)$$

and using the fact that Φ and Ψ are constant on the horizon, the variation of I_2 may be evaluated giving

$$\delta_A I_2 = Q\delta\Phi_H - \Psi_H\delta P + \delta\Omega_H I_3. \quad (5.15)$$

Substituting this into equation (5.1) shows that there is now no need to evaluate the final integral I_3 since it cancels in (5.1) and the First Law of Black Hole Mechanics for stationary axi-symmetric black holes in non-linear electrodynamics has been established:

$$\delta M = \frac{\kappa}{8\pi} \delta\mathcal{A} + \Omega_H \delta J + \Phi_H \delta Q + \Psi_H \delta P. \quad (5.16)$$

This agrees with the known formula for Kerr-Newman black holes in Einstein-Maxwell theory, which is of course a special case of the above calculation.

It is interesting to note that the final term $\delta\Omega_H I_3$ in (5.1), which would have given a contribution $(J - J_H)\delta\Omega_H$ due to the angular momentum of the electromagnetic field, cancelled with part of the previous term $\delta_A I_2$. This is a result of the fact that *co-rotating* electric and magnetic fields were used in the calculation (in order that their potentials be constant on the horizon). Thus the rotational effects of the electromagnetic field are automatically taken into account in the last two terms of (5.16).

6 Born-Infeld Theory

The one non-linear theory of electrodynamics which keeps making appearances again and again in many different contexts within modern theoretical physics is Born-Infeld theory. Amongst its many special properties is an exact $SO(2)$ electric-magnetic duality invariance. The Lagrangian density describing Born-Infeld theory (in arbitrary spacetime dimensions) is

$$\mathfrak{L}_F = \sqrt{g} L_F = \frac{4}{b^2} \left\{ \sqrt{g} - \sqrt{|\det(g_{\mu\nu} + bF_{\mu\nu})|} \right\} \quad (6.1)$$

where b is a fundamental parameter of the theory, with dimensions of mass. In open superstring theory, for example, loop calculations lead to this Lagrangian with $b = 2\pi\alpha'$. In four spacetime dimensions the determinant in (6.1) may be expanded out to give

$$L_F = \frac{4}{b^2} \left\{ 1 - \sqrt{1 + \frac{1}{2}b^2 F^2 - \frac{1}{16}b^4 (F \star F)^2} \right\} \quad (6.2)$$

which coincides with the usual Maxwell Lagrangian in the weak field limit.

The tensor $G^{\mu\nu}$ defined in section 2 is given by

$$G^{\mu\nu} = -\frac{1}{2} \frac{\partial L_F}{\partial F_{\mu\nu}} = \frac{F^{\mu\nu} - \frac{1}{4}b^2 (F \star F) \star F^{\mu\nu}}{\sqrt{1 + \frac{1}{2}b^2 F^2 - \frac{1}{16}b^4 (F \star F)^2}} \quad (6.3)$$

(so that $G^{\mu\nu} \approx F^{\mu\nu}$ for weak fields) and satisfies the electromagnetic equations of motion

$$\nabla_\mu G^{\mu\nu} = 0 \quad (6.4)$$

which are highly non-linear in $F_{\mu\nu}$. The energy-momentum tensor may be written as

$$T_{\mu\nu} = \frac{F_\mu{}^\lambda F_{\nu\lambda} + \frac{1}{b^2} \left[\sqrt{1 + \frac{1}{2}b^2 F^2 - \frac{1}{16}b^4 (F \star F)^2} - 1 - \frac{1}{2}b^2 F^2 \right] g_{\mu\nu}}{\sqrt{1 + \frac{1}{2}b^2 F^2 - \frac{1}{16}b^4 (F \star F)^2}}. \quad (6.5)$$

Although it is by no means obvious, it may be verified that equations (6.3)–(6.5) are invariant under electric-magnetic duality: $F \leftrightarrow \star G$.

In flat space, and for purely electric configurations, the Lagrangian (6.2) reduces to

$$L_F = \frac{4}{b^2} \left\{ 1 - \sqrt{1 - b^2 \mathbf{E}^2} \right\} \quad (6.6)$$

so there is an upper bound on the electric field strength \mathbf{E} :

$$|\mathbf{E}| \leq \frac{1}{b}. \quad (6.7)$$

The field due to a point charge is

$$E_r = \frac{Q}{\sqrt{r^4 + b^2 Q^2}} \quad (6.8)$$

and so achieves the bound (6.7) at $r = 0$. The total self-energy of the point charge is thus

$$\mathcal{E} = \frac{1}{4\pi} \int d^3\mathbf{x} T_{00} = \frac{1}{4\pi} \int d^3\mathbf{x} \frac{1}{b^2 r^2} \left(\sqrt{r^4 + b^2 Q^2} - r^2 \right). \quad (6.9)$$

Integrating by parts gives a standard elliptic integral:

$$\mathcal{E} = \frac{2Q^2}{3} \int_0^\infty \frac{dr}{\sqrt{r^4 + b^2 Q^2}} = \frac{(\pi Q)^{\frac{3}{2}}}{3\sqrt{b} \Gamma\left(\frac{3}{4}\right)^2} \quad (6.10)$$

which is finite (for simplicity, Q and b are taken to be positive here). Thus Born-Infeld theory succeeded in its original goal of providing a model for point charges with finite self-energy. Note that in the limit $b \rightarrow 0$, Maxwell theory is reproduced and the self-energy diverges.

Now consider static spherically symmetric black holes in this theory. Using electric-magnetic duality, there is no loss of generality in considering only electrically charged black holes. The solution is

$$ds^2 = - \left(1 - \frac{2m(r)}{r} \right) dt^2 + \left(1 - \frac{2m(r)}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

$$G_{tr} = \frac{Q}{r^2} \quad \text{or} \quad F_{tr} = \frac{Q}{\sqrt{r^4 + b^2 Q^2}}, \quad (6.11)$$

where the function $m(r)$ satisfies

$$m'(r) = \frac{1}{b^2} \left(\sqrt{r^4 + b^2 Q^2} - r^2 \right) \quad (6.12)$$

and $'$ denotes differentiation with respect to r . The mass M is given by

$$M = \lim_{r \rightarrow \infty} m(r) \quad (6.13)$$

and hence

$$m(r) = M - \frac{1}{b^2} \int_r^\infty dx \left(\sqrt{x^4 + b^2 Q^2} - x^2 \right), \quad (6.14)$$

which is a monotonically increasing function of r . The horizons are given by the roots of the equation $r = 2m(r)$ and so the number of horizons will be determined by $m(0)$ and $m'(0)$. $m(0)$ depends on the self-energy of the

electromagnetic field, the integral being the same as for the point charge in flat space:

$$m(0) = M - \frac{(\pi Q)^{\frac{3}{2}}}{3\sqrt{b}\Gamma\left(\frac{3}{4}\right)^2} = M - \mathcal{E} \quad (6.15)$$

and so $m(0)$ may be interpreted as the binding energy. From (6.12) one has

$$m'(0) = \frac{Q}{b}. \quad (6.16)$$

For $m(0) > 0$ there is precisely one non-degenerate horizon. If $m(0) = 0$ then there is one non-degenerate horizon for $Q > \frac{1}{2}b$ and none otherwise. The case $m(0) < 0$ is similar to Reissner-Nordström, with either no horizons, one degenerate horizon or two non-degenerate horizons, depending on the relative magnitudes of M , Q and b . Note that the Reissner-Nordström solution is recovered in the limit $b \rightarrow 0$ in which case $m(0) \rightarrow -\infty$.

Assuming that there is at least one horizon, let r_+ denote the outer event horizon. It has surface gravity κ given by

$$\kappa = -\frac{1}{2} \left. \frac{dg_{tt}}{dr} \right|_{r=r_+} = \frac{1}{2r_+} - \frac{1}{b^2 r_+} \left(\sqrt{r_+^4 + b^2 Q^2} - r_+^2 \right) \quad (6.17)$$

and a surface area \mathcal{A} of

$$\mathcal{A} = 4\pi r_+^2. \quad (6.18)$$

The electric potential on the horizon (in a gauge in which it vanishes at infinity) is

$$\Phi_H = \int_{r_+}^{\infty} dr \frac{Q}{\sqrt{r^4 + b^2 Q^2}} \quad (6.19)$$

and it is simple to verify that the First Law

$$dM = \frac{\kappa}{8\pi} d\mathcal{A} + \Phi_H dQ \quad (6.20)$$

is indeed satisfied. On the other hand, the usual statement of Smarr's formula (1.2) does not hold.

7 Inclusion of Scalar Fields

In all the cases in which a non-linear theory of electrodynamics arises from string theory or D-brane theories, there are always additional scalar fields present. In general, these fields are coupled to the electromagnetic field in some non-trivial way. It is thus interesting to ask in what ways the presence of scalars affects the results obtained above. The simplest case would be to consider a single scalar field. For example, Einstein-Maxwell-dilaton theory which may be described by the Lagrangian

$$L_F = -2 (\nabla\phi)^2 - e^{-2\alpha\phi} F^2. \quad (7.1)$$

In the case $\alpha = 1$ this coincides with a certain limit of string theory and $\alpha = -\sqrt{3}$ gives Kaluza-Klein theory. One may also include an axion. For example, the Lagrangian

$$L_F = -2 (\nabla\phi)^2 - 2e^{4\phi} (\nabla a)^2 - e^{-2\phi} F^2 + 2a F \star F \quad (7.2)$$

describes the bosonic sector of $N = 4$ supergravity in 4 dimensions and also string theory compactified on a torus. The equations of motion derived from the above Lagrangian have an $SL(2, \mathbb{R})$ invariance which includes simple electric-magnetic duality as a subgroup. Of course, there may also be higher order corrections giving non-linear electrodynamics coupled to an axion and a dilaton. In this case, under certain circumstances, the theory may retain its $SL(2, \mathbb{R})$ duality [8]. This is thus of interest in string theory where the duality is S-duality, relating the strong and weak coupling limits of the theory.

To try to extend the results of this paper to include theories of the above type, consider the Lagrangian

$$L_F = -2\mathcal{G}_{ab}(\phi^c)(\nabla_\mu\phi^a)(\nabla^\mu\phi^b) + \tilde{L}_F(F_{\mu\nu}, \phi^a). \quad (7.3)$$

Here \mathcal{G}_{ab} is the (symmetric) metric on the target space of the scalar fields and \tilde{L}_F is a function of the scalars ϕ^a as well as $F_{\mu\nu}$ but contains no derivative terms. Clearly then, the two Lagrangians (7.1) and (7.2) above are special cases of this one. The tensor $G^{\mu\nu}$ is defined in the same way as in section 2:

$$G^{\mu\nu} = -\frac{1}{2} \frac{\partial L_F}{\partial F_{\mu\nu}} = -\frac{1}{2} \frac{\partial \tilde{L}_F}{\partial F_{\mu\nu}} \quad (7.4)$$

and so now it will, in general, depend on the scalar fields as well as $F_{\mu\nu}$. The electromagnetic equations of motion, obtained by varying with respect to A_μ , are again

$$\nabla_\mu G^{\mu\nu} = 0. \quad (7.5)$$

So the conserved electric and magnetic charges are defined as before. Varying the action with respect to ϕ^a gives the scalar equations of motion:

$$-4\nabla^\mu (\mathcal{G}_{ab} \nabla_\mu \phi^b) = \frac{\partial \tilde{L}_F}{\partial \phi^a} - 2 \frac{\partial \mathcal{G}_{bc}}{\partial \phi^a} (\nabla_\mu \phi^b) (\nabla^\mu \phi^c). \quad (7.6)$$

Varying with respect to the metric gives the Einstein field equations and the energy-momentum tensor now contains kinetic terms coming from the scalar fields:

$$T_{\mu\nu} = G_\mu{}^\lambda F_{\nu\lambda} + \mathcal{G}_{ab} (\nabla_\mu \phi^a) (\nabla_\nu \phi^b) + \frac{1}{4} g_{\mu\nu} L_F. \quad (7.7)$$

Considering stationary axi-symmetric solutions, the scalar fields must be time-translationally and rotationally invariant, $\mathcal{L}_k \phi^a = 0$ and $\mathcal{L}_m \phi^a = 0$. Hence

$$\mathcal{L}_\xi \phi^a = \xi^\mu \partial_\mu \phi^a = 0. \quad (7.8)$$

Therefore, contractions of $T_{\mu\nu}$ with ξ^μ are unaltered in form by the presence of the scalar fields and so the proofs that Φ_H , Ψ_H and κ are constant on the horizon \mathcal{H} follow as before. Also, for the same reason, the evaluation of the first and third integrals in (4.25) is unchanged. Thus the only change to the First Law comes from the second integral. This will contribute an additional term

$$-\frac{1}{16\pi} \delta_\phi \int_\Sigma dS_\mu \xi^\mu L_F \quad (7.9)$$

where δ_ϕ denotes the variation with respect to the scalar fields.

Using the scalar equations of motion, the variation of the Lagrangian L_F with respect to ϕ^a may be written as a total derivative:

$$\begin{aligned} \delta_\phi L_F &= -4\mathcal{G}_{ab} (\nabla_\mu \phi^a) (\nabla^\mu \delta \phi^b) + \left[\frac{\partial \tilde{L}_F}{\partial \phi^a} - 2 \frac{\partial \mathcal{G}_{bc}}{\partial \phi^a} (\nabla_\mu \phi^b) (\nabla^\mu \phi^c) \right] \delta \phi^a \\ &= -4\mathcal{G}_{ab} (\nabla_\mu \phi^a) (\nabla^\mu \delta \phi^b) - 4\nabla^\mu (\mathcal{G}_{ab} \nabla_\mu \phi^b) \delta \phi^a \\ &= -4\nabla^\mu (\mathcal{G}_{ab} (\nabla_\mu \phi^a) \delta \phi^b). \end{aligned} \quad (7.10)$$

Thus the variation of the integrand in (7.9) with respect to ϕ is also a total derivative:

$$\xi^\mu \delta_\phi L_F = -8 \nabla_\nu \left(\xi^{[\mu} \mathcal{G}_{ab} (\nabla^{\nu]} \phi^a) \delta \phi^b \right) \quad (7.11)$$

where use has been made of the fact that $\mathcal{L}_\xi(\mathcal{G}_{ab} \nabla^\mu \phi^a \delta \phi^b) = 0$. So, using Stoke's formula, the additional contribution (7.9) to the First Law may be written as surface integrals over the horizon \mathcal{H} and at spatial infinity. The contribution from the horizon vanishes identically using (7.8). Therefore the only additional contribution is an integral over spatial infinity:

$$\frac{1}{4\pi} \oint_\infty dS_{\mu\nu} \xi^\mu \mathcal{G}_{ab} (\nabla^\nu \phi^a) \delta \phi^b. \quad (7.12)$$

This integral is easily evaluated using the asymptotic form of the scalar fields at infinity

$$\phi^a \sim \phi_\infty^a + \frac{\Sigma^a}{r} + \mathcal{O}(r^{-2}) \quad (7.13)$$

where Σ^a are the scalar charges. The integral then gives $\mathcal{G}_{ab} \Sigma^a \delta \phi_\infty^b$ and so the modified First Law in the presence of scalar fields is

$$\delta M = \frac{\kappa}{8\pi} \delta \mathcal{A} + \Omega_H \delta J + \Phi_H \delta Q + \Psi_H \delta P + \mathcal{G}_{ab} \Sigma^a \delta \phi_\infty^b. \quad (7.14)$$

This agrees with the result obtained in [17]. In that case allowing the asymptotic values of the scalar fields to vary was of relevance to string theory where different asymptotic values label different vacua of the theory.

Of course one may also consider more than one $U(1)$ gauge field and the above result generalizes in the obvious way. This is relevant to the bosonic sector of supergravity theories in 4 dimensions for example and also string theories compactified down to 4 dimensions. In fact the Lagrangian (7.3) is general enough to cover a very wide range of theories. The statement of the First Law (7.14) is consistent with the “no hair” theorems, indicating that the black holes of such theories are completely specified by their mass, angular momentum, electric and magnetic charges and the asymptotic values of the scalars. The scalar charges are then determined uniquely.

It was remarked above that, because the First Law does hold for non-linear electrodynamics, the Smarr formula does not. To make this more explicit, consider first the case of *linear* electrodynamics. In this case the

Lagrangian must be quadratic in $F_{\mu\nu}$. So, considering N $U(1)$ gauge fields labelled by $\Lambda, \Sigma = 1, 2, \dots, N$, the Lagrangian must take the form

$$L_F = -2\mathcal{G}_{ab}(\nabla_\mu\phi^a)(\nabla^\mu\phi^b) - \mu_{\Lambda\Sigma}F_{\mu\nu}^\Lambda F^{\Sigma\mu\nu} + \nu_{\Lambda\Sigma}F_{\mu\nu}^\Lambda \star F^{\Sigma\mu\nu} \quad (7.15)$$

where $\mu_{\Lambda\Sigma}$ and $\nu_{\Lambda\Sigma}$ are arbitrary functions of the scalar fields ϕ^a . This is precisely the class of theories discussed in [17]. Now, using the no hair theorem, the mass may be regarded as a function of the horizon area \mathcal{A} , the angular momentum J , the electric and magnetic charges Q^Λ , P^Λ and the asymptotic values of the scalar fields ϕ_∞^a . Furthermore, on dimensional grounds, it is clear that any other parameters of the theory must be dimensionless² and so M is a homogeneous function. Using the fact that \mathcal{A} and J have units of $(\text{mass})^2$, Q^Λ and P^Λ have units of mass and ϕ_∞^a is dimensionless, Euler's theorem for homogeneous functions implies that

$$M = 2\mathcal{A}\frac{\partial M}{\partial \mathcal{A}} + 2J\frac{\partial M}{\partial J} + Q^\Lambda\frac{\partial M}{\partial Q^\Lambda} + P^\Lambda\frac{\partial M}{\partial P^\Lambda}. \quad (7.16)$$

The partial derivatives are given by the First Law (7.14) and thus one obtains Smarr's formula

$$M = \frac{\kappa\mathcal{A}}{4\pi} + 2\Omega_H J + \Phi_\Lambda Q^\Lambda + \Psi_\Lambda P^\Lambda. \quad (7.17)$$

Note that its form is unaltered by the presence of the scalar fields as was pointed out in [17].

Now consider non-linear theories of electrodynamics. For example, consider the higher order corrections which result from loop calculations in string theory, leading to Born-Infeld-type Lagrangians. Since the Lagrangian is no longer quadratic in $F_{\mu\nu}$, one may argue on dimensional grounds that there will always be a coupling constant with non-trivial mass dimensions (for example, the constant b in Born-Infeld theory discussed in the last section). In general this coupling constant will enter into the formula for the mass and so M will no longer be a homogeneous function of \mathcal{A} , J , Q^Λ , P^Λ and ϕ_∞^a . Thus Smarr's formula will no longer hold but, as one might have expected, the First Law remains true, i.e. the black hole's thermodynamics are preserved by higher order quantum corrections.

²recall $G = c = 4\pi\epsilon_0 = 1$ in this paper, so the dimensions of any quantity may be expressed in terms of mass dimensions

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References

- [1] M. Born and L. Infeld, Proc. Roy. Soc. (Lond.) **A144**, 425 (1934)
M. Born, Ann. Inst. Poincaré **7**, 155 (1939)
- [2] E. Fradkin and A. Tseytlin, Phys. Lett. **B163**, 123 (1985)
E. Bergshoeff, E. Sezgin, C. Pope and P. Townsend, Phys. Lett. **B188**, 70 (1987)
R. Metsaev, M. Rahmanov and A. Tseytlin, Phys. Lett. **B193**, 207 (1987)
C. Callan, C. Lovelace, C. Nappi and S. Yost, Nucl. Phys. **B308**, 221 (1988)
O. Andreev and A. Tseytlin, Nucl. Phys. **B311**, 205 (1988)
- [3] R. Leigh, Mod. Phys. Lett. **A4**, 2767 (1989)
- [4] J. Schwinger, Phys. Rev. **82**, 664 (1951)
W. Heisenberg and H. Euler, Z. Phys **98**, 714 (1936)
- [5] C. Bachas and M. Porrati, Phys. Lett. **B296**, 77 (1992)
C. Bachas, hep-th/9303063
- [6] E. Schrödinger, Proc. Roy. Soc. (Lond.) **A150**, 465 (1935)
- [7] G. Gibbons and D. Rasheed, Nucl. Phys. **B454**, 185 (1995)
- [8] G. Gibbons and D. Rasheed, Phys. Lett. **B365**, 46 (1996)
- [9] H. Soleng, CERN-TH/95-110
- [10] D. Palatnik, quant-ph/9701017

- [11] J. Bardeen, B. Carter and S. Hawking, Comm. Math. Phys. **31**, 161 (1973)
 B. Carter in *General Relativity: An Einstein Centenary Survey*, ed. S. Hawking and W. Israel, Cambridge University Press (1979)
 B. Carter in *Black Holes*, ed. C. and B. DeWitt (Les Houches Summer School Lectures 1972), New York: Gordon and Breach
- [12] M. Heusler and N. Straumann, Class. Quant. Grav. **10**, 1299 (1993)
- [13] For excellent reviews, see also:
 R. Wald, *General Relativity*, University of Chicago Press (1984)
 M. Heusler, *Black Hole Uniqueness Theorems*, Cambridge University Press (1996)
- [14] R. Wald, gr-qc/9305022
 D. Sudarsky and R. Wald, Phys. Rev. **D46**, 1453 (1990)
- [15] G. Gibbons and M. Perry, Proc. Roy. Soc. (Lond.) **A358**, 467 (1978)
 [in *Euclidean Quantum Gravity*, ed. G. Gibbons and S. Hawking, World Scientific (1993)]
- [16] B. Kay and R. Wald, Phys. Rept. **207**, 49 (1991)
- [17] G. Gibbons, R. Kallosh and B. Kol, Phys. Rev. Lett. **77**, 4992 (1996)